Exercise Sheet

for Lecture *Quantum-Information, -Computing, and -Sensing*

Stefan Scheel, Friedemann Reinhard, Clemens Cap, Arman Kashef

5th July 2024

1. Error Probability

Goal: We want to learn how to work with the channel matrix and to interpret the per-pair transinformation.

We study the channel from the lecture with $A = \{R, S\}$, $B = \{\rho, \sigma, \tau\}$ and the channel matrix

$$\begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}$$

We connect the input of the channel to a source over the set $\{R, S\}$ without further encoding or source extension. We connect the output of the channel to the decoding function $\rho \mapsto R, \sigma \mapsto S, \tau \mapsto S$.

Task 1: What is the error probability of the resulting communication system, when the source has the probability distribution $p(R) = p(S) = 0.5$.

Task 2: For which probability distribution $p(R)$ and $p(S)$ is the error probability minimal?

Task 3: Interpret the result of task 2. Explain why it is reasonable that the Shannon channel coding theorem deals with rate and error in a combined way.

Task 4: Determine the probabilities for the compound which we obtain when the source has the probability distribution $p(R) = r$ and $p(S) = 1 - r$.

Task 5: Determine the per-pair transinformation for varying r, analyze its sign and interpret the results.

2. Capacity of a Channel

Goal: We want to determine the capacity of an asymmetric binary channel and study the channel equation.

We have a channel with input alphabet $\{A, B\}$, output alphabet $\{C, D\}$ and channel matrix

$$\begin{pmatrix} 2/3 & 1/3 \\ 0 & 1 \end{pmatrix}$$

We connect this channel to an input source with $p(A) = r$ and $p(B) = 1 - r$.

Task 1: For which value of r do we get an optimal transmission of information from the input source to the output port?

Task 2: What is the capacity of the channel?

Task 3: When connected to the optimal channel, which amount of noise is generated by the channel, which amount of information is destroyed by the channel and which amount of information is transported from the input port to the output port.

3. And Gate Information Destruction

Goal: We want to illustrate the information loss along an and gate.

In the lecture it was shown that the and-gate destroys information in the following sense: We consider the input as two independent, binary sources, which both are equidistributed. We consider the output as a single binary source. Then the entropy at the input ports is larger than the entropy at the output port.

Task 1: Is it possible to find distribution(s) for the two input sources such that there is no information loss at all?

Task 2: For which distribution(s) of the two input sources is the information loss maximal? How large is it in this case?

Task 3: Are there distributions for the two input sources, such that the information loss along the gate is exactly one bit?

Hint for all the three exercises: The equations in information theory often cannot be solved in closed form. Thus the use of Mathematica, Matlab, Jupyter, GnuOctave, GnuPlot, PICTEXor similar systems for visualization and computation may be helpful.